CH 24 – RADICALS, PART I

❑ INTRODUCTION

Berkeley Radical

$a^2 + b^2 = c^2$

We saw square roots when we studied the Pythagorean Theorem. They may have been hidden, but when the end of a right triangle problem resulted in an equation like $c^2 = 144$, we used the notion of *square root* when we determined that c had to be 12. (Another solution to this equation is $c = -12$, but it would be discarded in a triangle problem.) And you may recall the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ from earlier classes, where the square root is clearly seen.

❑ THE MEANING OF SQUARE ROOTS

> Every positive number has <u>two</u> square roots.

Fact: The number 81 has <u>two</u> square roots, 9 and –9. This is because $9^2 = 81$, and also $(-9)^2 = 81$.

Because 11^2 and $(-11)^2$ both equal 121, the two square roots of 121 are 11 and –11.

The number 1 has two square roots, namely 1 and –1. The reason: If either 1 or –1 is squared, the result is 1.

Zero has just one square root, 0, since $0^2 = 0$, and no other number squared would result in 0.

The number -16 has <u>no</u> square roots in the real numbers. After all, what real number squared equals -16? 4 can't work, since $4^2 = 16$. Nor can -4 work, since $(-4)^2$ is also 16. In fact, any real number squared produces an answer that is at least zero. That is, there is no real number that can be squared and result in -16. Thus, -16 has no square roots in $\mathbb{R}$ (the set of real numbers).

Consider the fraction $\frac{1}{4}$. It has two square roots, $\frac{1}{2}$ and $-\frac{1}{2}$. Why? Because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$.

We can encapsulate this section by stating that

a is a square root of b if $a^2 = b$

Homework

1.
   a. Find as many numbers as you can whose square is 144.

   b. Find as many numbers as you can whose square is -144.

   c. How many square roots does 25 have? What are they? Why are they square roots of 25?

   d. How many square roots does 196 have? What are they? Why are they square roots of 196?

   e. How many square roots does 1 have? What are they?

   f. How many square roots does 0 have? What are they?

   g. How many square roots does -9 have? What are they?

   h. What are the two square roots of $\frac{4}{9}$?

❑ NOTATION

The previous section showed us that the number 81 actually has two square roots, 9 and –9. How do we denote these two square roots of 81 without actually calculating their final values? Notice that 81 has a positive square root (9) and a negative square root (–9). We write

$\sqrt{81} = 9$ The positive square root of 81 is 9

and $-\sqrt{81} = -9$ The negative square root of 81 is –9

That is, the positive square root is written with a radical sign, while the negative square root is written with a radical sign preceded by a minus sign.

The number 121 has two square roots, which we can write as $\sqrt{121} = 11$ and $-\sqrt{121} = -11$.

Thus, if someone asks you for the square root of 100, you must answer 10 <u>and</u> –10, because each of these is a square root of 100. However, if you're presented with "$\sqrt{100}$", you answer just 10, <u>not</u> –10, because we agreed that the radical sign refers to the non-negative square root only. Also, if someone asks you for the "positive" or "principal" square root of 100, then you must answer <u>just 10</u>, not –10.

The square root symbol $\sqrt{}$ is called the ***radical sign***, and the quantity inside the radical sign is called the ***radicand***. Why do we need a radical sign at all? After all, the two square roots of 49 are 7 and –7; it's just that simple. But what about the number 17? It also has two square roots, although, as we've seen before, they don't result in nice numbers. So we must have some notation to denote the square roots of 17 even though we might not care to work them out. So we use the radical sign and state that

There are two square roots of 17: $\sqrt{17}$ and $-\sqrt{17}$.

The important thing to note is that $\left(\sqrt{17}\right)^2 = 17$ and $\left(-\sqrt{17}\right)^2 = 17$.

We make the following four conclusions:

- Every positive number has two square roots, one positive and one negative. The two square roots of n are denoted $\sqrt{n}$ and $-\sqrt{n}$.

- Zero has one square root (namely itself) — $\sqrt{0} = 0$.

- Negative numbers don't have any square roots at all (at least in this course). For example, $\sqrt{-9}$ does not exist.

- The most important issue in this whole discussion is that, *assuming x is positive*:

The symbol $\sqrt{x}$ means the positive square root only!

Homework

2. Use your calculator to give some credence to the statement that $(\sqrt{13})^2 = 13$.

3. a. How many square roots does 37 have?

   b. What are they?

   c. Prove that they are square roots of 37.

4. Evaluate each expression:

a. $\sqrt{144}$ b. $-\sqrt{225}$ c. $\sqrt{-25}$ d. $-\sqrt{-36}$

e. $\sqrt{196}$ f. $\sqrt{1}$ g. $\sqrt{0}$ h. $\sqrt{256}$

i. $-\sqrt{64}$ j. $\sqrt{169}$ k. $\sqrt{-100}$ l. $\sqrt{289}$

5. Simplify each expression:

a. $(\sqrt{64})^2$ b. $(\sqrt{93})^2$ c. $(\sqrt{a})^2$ d. $(\sqrt{m})^2$

6. Consider the following examples:

i) $\sqrt{9^2} = \sqrt{81} = 9$ ii) $\sqrt{(-6)^2} = \sqrt{36} = 6$

Now work the following problems (without a calculator):

a. $\sqrt{10^2}$ b. $\sqrt{12{,}456^2}$ c. $\sqrt{(-7)^2}$ d. $\sqrt{(-3)^2}$

7. It's easy to confuse the phrases "the negative square root of a number" and "the square root of a negative number." Give examples which clarify the difference between these phrases.

❑ *SIMPLIFYING SQUARE ROOTS*

EXAMPLE 1: **Simplify: $\sqrt{50}$**

Solution:

We start with: $\sqrt{50}$

Factor 50 into 25 times 2: $\sqrt{25 \cdot 2}$

Split the radical down the middle: $\sqrt{25} \cdot \sqrt{2}$

Now calculate the square root: $5 \cdot \sqrt{2}$

And remove the dot: $\boxed{5\sqrt{2}}$

You might wonder how we knew to factor 50 as $25 \cdot 2$ and not as $10 \cdot 5$, perhaps. Well, we could have, but it wouldn't have done us any good. This is because neither the 10 nor the 5 has a nice square root, whereas the 25 does.

We can generalize the previous example by saying that "*The square root of a product is equal to the product of the square roots*." Thus, assuming that a and b are both non-negative,

$$\boxed{\sqrt{ab} = \sqrt{a}\sqrt{b}}$$

<u>EXAMPLE 2:</u> **Simplify each square root:**

A. $\sqrt{144} = \sqrt{36 \cdot 4} = \sqrt{36} \cdot \sqrt{4} = 6 \cdot 2 = \mathbf{12}$
(Of course, the steps aren't needed, since $\sqrt{144}$ is easily seen to be 12, but it proves again that the method works.)

B. $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = \mathbf{5\sqrt{3}}$

C. $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = \mathbf{2\sqrt{10}}$

D. $\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = \mathbf{3\sqrt{5}}$

E. $-\sqrt{288} = -\sqrt{144 \cdot 2} = -\sqrt{144} \cdot \sqrt{2} = \mathbf{-12\sqrt{2}}$

F. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = \mathbf{6\sqrt{2}}$

This problem could have been done in more steps; you are encouraged to use as many steps as you see fit (unless your instructor thinks otherwise). For example,

$$\sqrt{72} = \sqrt{9 \cdot 8} = \sqrt{9} \cdot \sqrt{8} = 3\sqrt{8} = 3\sqrt{4 \cdot 2}$$

$$= 3 \cdot \sqrt{4} \cdot \sqrt{2} = 3 \cdot 2 \cdot \sqrt{2} = \mathbf{6\sqrt{2}}$$

It's important to understand that if you had stopped at the step $3\sqrt{8}$, you would not have the right answer, even though you simplified the radical somewhat. This is because even more factors could be brought out of the square root sign.

G. $\sqrt{30}$ cannot be simplified. 30 does not contain any factor which has a nice square root.

H. $\sqrt{-64}$ is not a real number, so there is nothing to simplify.

Homework

8. In part F of the above example we concluded that $\sqrt{72} = 6\sqrt{2}$. Use your calculator to "verify" this fact.

9. Simplify each square root:

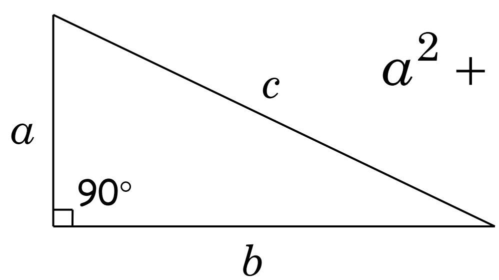
a. $\sqrt{8}$	b. $\sqrt{24}$	c. $\sqrt{-75}$	d. $\sqrt{27}$	e. $\sqrt{98}$
f. $\sqrt{54}$	g. $\sqrt{150}$	h. $\sqrt{172}$	i. $\sqrt{105}$	j. $\sqrt{-20}$
k. $\sqrt{175}$	l. $\sqrt{200}$	m. $\sqrt{242}$	n. $\sqrt{46}$	o. $\sqrt{-1}$
p. $\sqrt{1}$	q. $\sqrt{0}$	r. $\sqrt{88}$	s. $\sqrt{147}$	t. $\sqrt{361}$
u. $\sqrt{338}$	v. $\sqrt{147}$	w. $\sqrt{250}$	x. $\sqrt{184}$	y. $\sqrt{189}$

10. Simplify each square root:

a. $\sqrt{18}$	b. $-\sqrt{72}$	c. $\sqrt{12}$	d. $\sqrt{32}$	e. $-\sqrt{52}$
f. $\sqrt{162}$	g. $\sqrt{125}$	h. $\sqrt{128}$	i. $\sqrt{256}$	j. $\sqrt{28}$
k. $\sqrt{-1}$	l. $-\sqrt{500}$	m. $\sqrt{0}$	n. $\sqrt{625}$	o. $\sqrt{800}$
p. $\sqrt{600}$	q. $\sqrt{-4}$	r. $\sqrt{338}$	s. $-\sqrt{121}$	t. $\sqrt{450}$
u. $\sqrt{363}$	v. $\sqrt{117}$	w. $-\sqrt{350}$	x. $\sqrt{490}$	y. $\sqrt{396}$

❑ *SOLVING RIGHT TRIANGLES*

Recall the Pythagorean Theorem:



$$a^2 + b^2 = c^2$$

<u>EXAMPLE 3:</u> **Solve each right triangle problem:**

A. **The legs are 6 and 10. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{6}^2 + \mathbf{10}^2 = c^2 \Rightarrow 36 + 100 = c^2$$
$$\Rightarrow c^2 = 136 \Rightarrow c = \sqrt{136} = \sqrt{4 \cdot 34} = \mathbf{2\sqrt{34}}$$

Notice that $c^2 = 136$ is a quadratic equation, and therefore probably has two solutions; in fact, the solutions are $\pm\sqrt{136}$, or $\pm 2\sqrt{34}$. But in this problem we're talking about the hypotenuse of a triangle, whose length must be positive. So we immediately discard the negative solution and retain just the positive one.

B. **One leg is 20 and the hypotenuse is 30. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{20}^2 + b^2 = \mathbf{30}^2 \Rightarrow 400 + b^2 = 900$$

$$\Rightarrow b^2 = 500 \Rightarrow b = \sqrt{500} = \sqrt{100 \cdot 5} = \mathbf{10\sqrt{5}}$$

C. **The legs are $2\sqrt{7}$ and $3\sqrt{13}$. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow (\mathbf{2\sqrt{7}})^2 + (\mathbf{3\sqrt{13}})^2 = c^2$$

$$\Rightarrow 2^2\sqrt{7}^2 + 3^2\sqrt{13}^2 = c^2$$

$$\Rightarrow 4 \cdot 7 + 9 \cdot 13 = c^2$$

$$\Rightarrow 28 + 117 = c^2 \Rightarrow c^2 = 145 \Rightarrow c = \sqrt{145}$$

Notice the use of parentheses around the legs. Also notice how we have used one of the law of exponents: $(ab)^n = a^n b^n$.

D. **The hypotenuse is $\sqrt{72}$ and one of the legs is 8. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow a^2 + \mathbf{8}^2 = (\sqrt{\mathbf{72}})^2 \Rightarrow a^2 + 64 = 72$$

$$\Rightarrow a^2 = 8 \Rightarrow a = \sqrt{8} = \sqrt{4 \cdot 2} = \mathbf{2\sqrt{2}}$$

E. **One of the legs is $3\sqrt{2}$ and the hypotenuse is $\sqrt{42}$. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow a^2 + (\mathbf{3\sqrt{2}})^2 = (\sqrt{\mathbf{42}})^2$$

$$\Rightarrow a^2 + 18 = 42 \Rightarrow a^2 = 24 \Rightarrow a = \sqrt{24} = \mathbf{2\sqrt{6}}$$

Homework

Calculate the EXACT answer to each Pythagorean problem:

11. The legs of a right triangle are 8 and 12. Find the hypotenuse.
12. The legs of a right triangle are 6 and 9. Find the hypotenuse.
13. Find the hypotenuse of a right triangle given that its legs have lengths of 5 and 7.
14. One leg of a right triangle is 15 and the hypotenuse is 20. Find the other leg.
15. The hypotenuse of a right triangle is 18 and one of its legs is 10. Find the other leg.
16. One leg of a right triangle is 5 and the hypotenuse is $\sqrt{43}$. Find the other leg.
17. Find the missing leg of a right triangle if the hypotenuse is $\sqrt{18}$ and one of its legs is 3.
18. The legs of a right triangle are $2\sqrt{3}$ and $3\sqrt{2}$. Find the hypotenuse.
19. The hypotenuse of a right triangle is $5\sqrt{7}$ and one of its legs is $2\sqrt{3}$. Find the length of the other leg.
20. A leg of a right triangle is $2\sqrt{8}$ and its hypotenuse is 10. Find the length of the other leg.
21. The leg of a right triangle is 5 and its hypotenuse is 4. Find the other leg.

❑ ALL KINDS OF ROOTS

Recap of Square Roots

The number 9 has two ***square roots***, 3 and –3. This is because $3^2 = 9$ and $(-3)^2 = 9$. The positive square root of 9 (the 3) is denoted $\sqrt{9}$, and the negative square root of 9 (the –3) is written $-\sqrt{9}$. In other words, $\sqrt{9} = 3$, and only 3, while $-\sqrt{9} = -3$.

In analyzing $\sqrt{-25}$, the positive square root of –25, we discover that we cannot find an answer for this problem, since the square of a real number can never be negative. If there is an answer to $\sqrt{-25}$, it lies outside $\mathbb{R}$, the set of real numbers.

Higher Roots

Consider the number 8. Since $2^3 = 8$, we can say that 2 is a ***cube root*** of 8. In fact, it's the only cube root of 8, simply because there's no other real number whose cube is 8. Perhaps a little surprising is that we can calculate the cube root of a negative number without leaving $\mathbb{R}$. For example, $\sqrt[3]{-27} = -3$, since $(-3)^3 = -27$.

The number 16 has two ***fourth roots***. The positive fourth root is $\sqrt[4]{16} = 2$, and the negative fourth root is $-\sqrt[4]{16} = -2$. After all, both 2 and –2 raised to the fourth power result in 16. However, just like square roots, $\sqrt[4]{-1}$ is not a real number.

The ***fifth root*** of 32 is 2; that is, $\sqrt[5]{32} = 2$. This is because $2^5 = 32$. Like cube roots, we can calculate the fifth root of a negative number. For example, $\sqrt[5]{-243}$ equals –3, since $(-3)^5 = -243$.

Homework

22. Find the square root(s) of

a. 100 b. 15 c. 0 d. –36 e. 1

23. Find the cube root(s) of

a. 64 b. –125 c. 0 d. 20 e. 1

24. Find the fourth root(s) of

a. 81 b. 0 c. –625 d. 25 e. 1

25. Find the fifth root(s) of

a. 1 b. 0 c. –243 d. 29 e. 32

26. Evaluate each expression:

a. $\sqrt{169}$ b. $\sqrt{225}$ c. $\sqrt[3]{8}$ d. $\sqrt[3]{27}$ e. $\sqrt[3]{-125}$

f. $\sqrt[4]{625}$ g. $\sqrt[4]{1}$ h. $\sqrt[4]{-16}$ i. $\sqrt[5]{-32}$ j. $\sqrt[5]{0}$

k. $\sqrt[3]{64}$ l. $\sqrt[3]{216}$ m. $\sqrt[3]{-64}$ n. $-\sqrt[5]{-1}$ o. $\sqrt[4]{16}$

p. $-\sqrt[4]{81}$ q. $\sqrt[3]{-1}$ r. $-\sqrt[4]{-1}$ s. $\sqrt[5]{243}$ t. $\sqrt{0}+\sqrt[3]{0}$

❑ *SIMPLIFYING MORE ROOTS*

Assume that x and y represent non-negative numbers. Then

$$\boxed{\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}}$$

The root of a product
is the product of the roots

Another useful rule for radicals is the following:

<u>If x is zero or positive</u>, then

$$\boxed{\sqrt[n]{x^n} = x}$$

The nth root cancels
the nth power

<u>EXAMPLE 4:</u> **Simplify each radical expression:**

A. $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = \mathbf{10\sqrt{2}}$

B. $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27} \times \sqrt[3]{3} = \mathbf{3\sqrt[3]{3}}$

C. $\sqrt[4]{1250} = \sqrt[4]{625 \cdot 2} = \sqrt[4]{625} \cdot \sqrt[4]{2} = \mathbf{5\sqrt[4]{2}}$

D. $\sqrt[3]{2250}$ Sometimes the radicand (the 2250) is too big to easily see if there's a perfect cube in it. So let's try a slightly different approach. We factor the 2250 into primes to get

$$2250 = 2 \cdot 3^2 \cdot 5^3$$

Clearly we can take the cube root of 5^3 (it's 5), but there are not enough of the other factors to take their cube roots. So we can write

$$\sqrt[3]{2250} = \sqrt[3]{2\cdot 3^2\cdot 5^3} = \sqrt[3]{5^3}\cdot\sqrt[3]{2\cdot 3^2} = \mathbf{5\sqrt[3]{18}}$$

Homework

27. Simplify each radical:

a. $\sqrt{288}$	b. $\sqrt[3]{54}$	c. $\sqrt[3]{16}$	d. $\sqrt[3]{250}$
e. $\sqrt[4]{32}$	f. $\sqrt[4]{243}$	g. $\sqrt[4]{162}$	h. $\sqrt[4]{1}$
i. $\sqrt[3]{-54}$	j. $\sqrt[4]{-16}$	k. $\sqrt[5]{64}$	l. $\sqrt[5]{486}$
m. $\sqrt[3]{135}$	n. $\sqrt[4]{162}$	o. $\sqrt[3]{189}$	p. $\sqrt[5]{96}$
q. $\sqrt[3]{128}$	r. $\sqrt[4]{1250}$	s. $\sqrt[3]{250}$	t. $\sqrt[3]{432}$
u. $\sqrt[5]{320}$	v. $\sqrt[3]{48}$	w. $\sqrt[4]{648}$	x. $\sqrt[5]{2673}$

❑ *THE SQUARE ROOT OF A SQUARE*

Our goal in this section is to analyze the expression

$$\sqrt{x^2}$$

Although it seems that this expression is simply x, we'll soon see that this is not necessarily the case. Our modus operandi (method of operation) for this problem will be graphing. We'll graph the formula

$$y = \sqrt{x^2}$$

by plotting lots of points, and then see what we can see. We'll do three calculations together, and then you do the rest.

If $x = 4$,

then $y = \sqrt{x^2} = \sqrt{4^2} = \sqrt{16} = 4$, so that **(4, 4)** is on the graph.

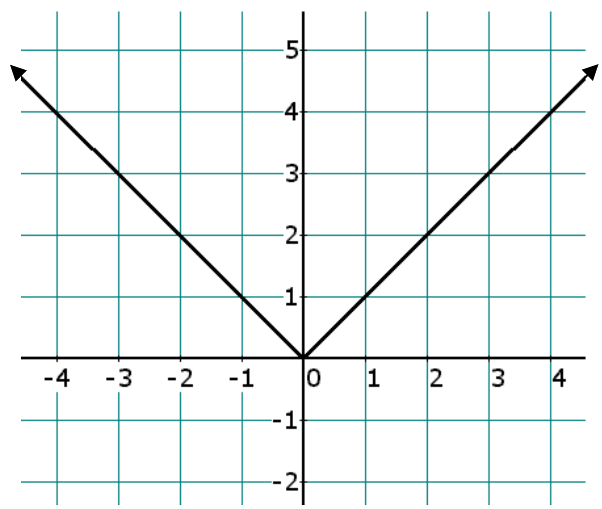
Let $x = 0$;

then $y = \sqrt{x^2} = \sqrt{0^2} = \sqrt{0} = 0$. Thus, our graph contains the origin **(0, 0)**.

Choose $x = -3$;

this choice of x gives $y = \sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$, so that **(–3, 3)** is a point on the graph. [Strange . . . x was negative 3, but y came out positive 3.] Continuing with more points, and then the plotting of those points, gives us the following graph:

x	y
–4	4
–3	3
–2	2
–1	1
0	0
1	1
2	2
3	3
4	4



Does this graph look familiar? It's from way back in Chapters 2 and 3, and it's the graph of the **absolute value** function, $y = |x|$. In other words, since we graphed $y = \sqrt{x^2}$ and ended up with $y = |x|$, we can conclude that

$$\boxed{\sqrt{x^2} = |x|}$$

So when you see something like $\sqrt{x^2}$, do NOT think that it simplifies to x — tempting, but wrong. Let's simplify a few expressions where we have the square root of a square:

$$\sqrt{h^2} = |h| \qquad \sqrt{(x+y+z)^2} = |x+y+z|$$

$$\sqrt{12^2} = |12| = 12 \qquad \sqrt{(-13)^2} = |-13| = 13$$

Note:

This issue with square roots and absolute values does <u>not</u> arise with cube roots. For example,

$$\sqrt[3]{5^3} = \sqrt[3]{125} = 5 \qquad \text{and} \qquad \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$$

That is, whether you're taking the cube root of either a positive or a negative number cubed, the final answer is simply the number being cubed:

$\sqrt[3]{x^3} = x$, whether x is positive, negative, or zero

Homework

28. Simplify each radical expression:

a. $\sqrt{y^2}$ b. $\sqrt{(a-b)^2}$ c. $\sqrt{\pi^2}$ d. $\sqrt{(-99)^2}$

e. $\sqrt{(mn)^2}$ f. $\sqrt{(-\sqrt{2})^2}$ g. $\sqrt{0^2}$ h. $\sqrt{(a-b-c)^2}$

i. $\sqrt{(x+y)^2}$ j. $\sqrt{(1-t)^2}$ k. $\sqrt{(a+b)^2}$ l. $\sqrt{(u-w+z)^2}$

m. $\sqrt{(-123)^2}$ n. $\sqrt{-12^2}$ o. $\sqrt{a^2+b^2}$ p. $\sqrt{x^2-y^2}$

q. $\sqrt{x^2+6x+9}$ r. $\sqrt{25x^2-90x+81}$

Practice Problems

29. a. How many square roots does 169 have? What are they? Why are they square roots of 169?

    b. How many square roots does 29 have? What are they? Why are they square roots of 29?

    c. Name the only number with exactly one square root.

    d. Give a number which has no square root.

    e. What is the smallest possible value of the expression $\sqrt{x}$?

    f. Consider the number $-\sqrt{-36}$. Your uncle tells you that the answer is 6, since the negatives cancel out. Why is your uncle an ignoramus? Should you tell him that he is?

    g. Simplify: $\left(\sqrt{n}\right)^2$ (presuming n is a non-negative number.)

    h. Calculate $\sqrt{123{,}987{,}005^2}$ without a calculator.

    i. Calculate $\sqrt{(-123)^2}$ any way you like.

    j. Simplify: $\sqrt{x^2}$

    k. Simplify: $\sqrt{(a-b)^2}$

30. Simplify each square root:

a. $\sqrt{250}$	b. $\sqrt{56}$	c. $\sqrt{112}$	d. $\sqrt{400}$	e. $\sqrt{144}$
f. $\sqrt{76}$	g. $-\sqrt{288}$	h. $\sqrt{8}$	i. $\sqrt{4}$	j. $-\sqrt{54}$
k. $\sqrt{300}$	l. $\sqrt{9}$	m. $\sqrt{49}$	n. $\sqrt{196}$	o. $\sqrt{475}$
p. $\sqrt{72}$	q. $\sqrt{240}$	r. $\sqrt{100}$	s. $\sqrt{52}$	t. $\sqrt{96}$

u. $\sqrt{98}$ v. $-\sqrt{2}$ w. $\sqrt{648}$ x. $\sqrt{500}$ y. $\sqrt{-3}$

31. Simplify each square root:

a. $\sqrt{275}$ b. $\sqrt{931}$ c. $\sqrt{208}$ d. $\sqrt{343}$ e. $-\sqrt{63}$

f. $-\sqrt{175}$ g. $\sqrt{360}$ h. $\sqrt{44}$ i. $\sqrt{375}$ j. $\sqrt{68}$

k. $\sqrt{810}$ l. $\sqrt{80}$ m. $\sqrt{99}$ n. $\sqrt{252}$ o. $\sqrt{160}$

p. $\sqrt{153}$ q. $-\sqrt{180}$ r. $\sqrt{588}$ s. $\sqrt{135}$ t. $\sqrt{192}$

u. $\sqrt{425}$ v. $\sqrt{176}$ w. $\sqrt{92}$ x. $\sqrt{720}$ y. $\sqrt{-60}$

32. Simplify each radical:

a. $\sqrt[3]{108}$ b. $\sqrt[4]{405}$ c. $\sqrt[5]{192}$ d. $\sqrt[3]{-500}$ e. $\sqrt[4]{-32}$

f. $\sqrt[5]{-486}$ g. $\sqrt[3]{3000}$ h. $\sqrt[4]{567}$ i. $\sqrt[3]{648}$ j. $\sqrt[5]{320}$

k. $\sqrt[3]{81}$ l. $\sqrt[3]{250}$ m. $\sqrt[3]{-56}$ n. $\sqrt[4]{48}$ o. $\sqrt[4]{-405}$

p. $\sqrt[4]{768}$ q. $\sqrt[5]{128}$ r. $\sqrt[5]{1215}$

Solutions

1. a. 12 and –12 each have squares equal to 144.

b. There’s <u>no</u> number (in this class) whose square is –144.

c. 25 has two square roots, 5 and –5 (which we could write as ±5).
5 is a square root of 25 because $5^2 = 25$.
–5 is a square root of 25 because $(-5)^2 = 25$.

d. 196 has two square roots, 14 and -14.
14 is a square root of 196 because $14^2 = 196$.
-14 is a square root of 196 because $(-14)^2 = 196$.

e. 1 has two square roots, 1 and -1.

f. 0 has only one square root, 0.

g. -9 has no square root at all (in the real numbers).

h. The two square roots of $\frac{4}{9}$ are $\frac{2}{3}$ and $-\frac{2}{3}$.

2. $(\sqrt{13})^2 \approx (3.605551275)^2 \approx 13$

3. a. 2 b. $\sqrt{37}$ and $-\sqrt{37}$ c. $\left(\sqrt{37}\right)^2 = 37$ $\left(-\sqrt{37}\right)^2 = 37$

4. a. 12 b. -15 c. No answer d. No answer e. 14
f. 1 g. 0 h. 16 i. -8 j. 13
k. No answer l. 17

5. a. 64 b. 93 c. a d. m

6. a. 10 b. 12,456 c. 7 d. 3

7. An example of "the negative square root of a number" is $-\sqrt{100}$, which comes out to be -10, a fine answer.
As for "the square root of a negative number," an example would be $\sqrt{-1}$, which we've learned is not a real number,

8. We'll carry out each calculation to the 5th digit:

$$\sqrt{72} \approx 8.48528 \quad \text{and} \quad 6\sqrt{2} \approx 6(1.41421) = 8.48526$$

Did we get the same decimal result? Not exactly, but this was to be expected due to rounding. Nevertheless, this calculation gives pretty good evidence that $\sqrt{72}$ truly equals $6\sqrt{2}$. In fact, if you don't do any rounding at all, using all the digits your calculator can handle, the two quantities might have exactly the same digits.

9. a. $2\sqrt{2}$ b. $2\sqrt{6}$ c. Not real d. $3\sqrt{3}$ e. $7\sqrt{2}$
f. $3\sqrt{6}$ g. $5\sqrt{6}$ h. $2\sqrt{43}$ i. $\sqrt{105}$ j. Not real

k. $5\sqrt{7}$	l. $10\sqrt{2}$	m. $11\sqrt{2}$	n. $\sqrt{46}$	o. Not real
p. 1	q. 0	r. $2\sqrt{22}$	s. $7\sqrt{3}$	t. 19
u. $13\sqrt{2}$	v. $7\sqrt{3}$	w. $5\sqrt{10}$	x. $2\sqrt{46}$	y. $3\sqrt{21}$

10.

a. $3\sqrt{2}$	b. $-6\sqrt{2}$	c. $2\sqrt{3}$	d. $4\sqrt{2}$	e. $-2\sqrt{13}$
f. $9\sqrt{2}$	g. $5\sqrt{5}$	h. $8\sqrt{2}$	i. 16	j. $2\sqrt{7}$
k. Not real	l. $-10\sqrt{5}$	m. 0	n. 25	o. $20\sqrt{2}$
p. $10\sqrt{6}$	q. Not real	r. $13\sqrt{2}$	s. -11	t. $15\sqrt{2}$
u. $11\sqrt{3}$	v. $3\sqrt{13}$	w. $-5\sqrt{14}$	x. $7\sqrt{10}$	y. $6\sqrt{11}$

11. $4\sqrt{13}$ **12**. $3\sqrt{13}$ **13**. $\sqrt{74}$ **14**. $5\sqrt{7}$ **15**. $4\sqrt{14}$

16. $3\sqrt{2}$ **17**. 3 **18**. $\sqrt{30}$ **19**. $\sqrt{163}$ **20**. $2\sqrt{17}$

21. This scenario is impossible, since the hypotenuse of a right triangle must be longer than each of its legs. Here's a mathematical proof:

$$a^2 + b^2 = c^2$$

$$\Rightarrow a^2 + 5^2 = 4^2 \quad \text{(presuming that the leg is 5 and the hypotenuse is 4)}$$

$$\Rightarrow a^2 + 25 = 16$$

$$\Rightarrow a^2 = -9$$

$$\Rightarrow a = \sqrt{-9},$$

which is not a real number. And even if it becomes a number in a later class, you'll find that it still couldn't be considered the length of a side of a triangle.

22. a. ± 10 b. $\pm\sqrt{15}$ c. 0 d. Not real e. ± 1

23. a. 4 b. -5 c. 0 d. $\sqrt[3]{20}$ e. 1

24. a. ± 3 b. 0 c. Not real d. $\pm\sqrt[4]{25}$ e. ± 1

25. a. 1 b. 0 c. -3 d. $\sqrt[5]{29}$ e. 2

26.

a. 13	b. 15	c. 2	d. 3	e. -5
f. 5	g. 1	h. Not real	i. -2	j. 0

k. 4 l. 6 m. -4 n. 1 o. 2
p. -3 q. -1 r. Not real s. 3 t. 0

27. a. $12\sqrt{2}$ b. $3\sqrt[3]{2}$ c. $2\sqrt[3]{2}$ d. $5\sqrt[3]{2}$ e. $2\sqrt[4]{2}$ f. $3\sqrt[4]{3}$
g. $3\sqrt[4]{2}$ h. 1 i. $-3\sqrt[3]{2}$ j. Not real k. $2\sqrt[5]{2}$ l. $3\sqrt[5]{2}$
m. $3\sqrt[3]{5}$ n. $3\sqrt[4]{2}$ o. $3\sqrt[3]{7}$ p. $2\sqrt[5]{3}$ q. $4\sqrt[3]{2}$ r. $5\sqrt[4]{2}$
s. $5\sqrt[3]{2}$ t. $6\sqrt[3]{2}$ u. $2\sqrt[5]{10}$ v. $2\sqrt[3]{6}$ w. $3\sqrt[4]{8}$ x. $3\sqrt[5]{11}$

28. a. $|y|$ b. $|a-b|$ c. $|\pi| = \pi$ d. $|-99| = 99$
e. $|mn|$ f. $|-\sqrt{2}| = \sqrt{2}$ g. $|0| = 0$ h. $|a-b-c|$
i. $|x+y|$ j. $|1-t|$ k. $|a+b|$ l. $|u-w+z|$
m. 123 n. Not real o. As is p. As is
q. $|x+3|$ r. $|5x-9|$

29. a. 169 has two square roots, 13 and –13 (or ±13). They are square roots of 169 because $13^2 = 169$ and $(-13)^2 = 169$.
b. 29 has two square roots, $\sqrt{29}$ and $-\sqrt{29}$ (or $\pm\sqrt{29}$). They are square roots of 29 because $\left(\sqrt{29}\right)^2$ and $\left(-\sqrt{29}\right)^2$ equal 29.
c. 0; its only square root is 0.
d. –49, for instance, has no square root (in this class). In fact, no negative number has a square root in this class.
e. The smallest possible value is 0, and that occurs when $x = 0$.
f. The negatives do <u>not</u> cancel out; they're not adjacent to each other. The Order of Operations requires that we start with the square root, which doesn't exist because the radicand is negative. Should you inform your uncle that he's an ignoramus? Not if you might be in his will.
g. n
h. 123,987,005
i. 123
j. $|x|$
k. $|a-b|$

30. a. $5\sqrt{10}$ b. $2\sqrt{14}$ c. $4\sqrt{7}$ d. 20 e. 12
f. $2\sqrt{19}$ g. $-12\sqrt{2}$ h. $2\sqrt{2}$ i. 2 j. $-3\sqrt{6}$
k. $10\sqrt{3}$ l. 3 m. 7 n. 14 o. $5\sqrt{19}$
p. $6\sqrt{2}$ q. $4\sqrt{15}$ r. 10 s. $2\sqrt{13}$ t. $4\sqrt{6}$
u. $7\sqrt{2}$ v. $-\sqrt{2}$ w. $18\sqrt{2}$ x. $10\sqrt{5}$ y. Not real

31. a. $5\sqrt{11}$ b. $7\sqrt{19}$ c. $4\sqrt{13}$ d. $7\sqrt{7}$ e. $-3\sqrt{7}$
f. $-5\sqrt{7}$ g. $6\sqrt{10}$ h. $2\sqrt{11}$ i. $5\sqrt{15}$ j. $2\sqrt{17}$
k. $9\sqrt{10}$ l. $4\sqrt{5}$ m. $3\sqrt{11}$ n. $6\sqrt{7}$ o. $4\sqrt{10}$
p. $3\sqrt{17}$ q. $-6\sqrt{5}$ r. $14\sqrt{3}$ s. $3\sqrt{15}$ t. $8\sqrt{3}$
u. $5\sqrt{17}$ v. $4\sqrt{11}$ w. $2\sqrt{23}$ x. $12\sqrt{5}$ y. Not real

32. a. $3\sqrt[3]{4}$ b. $3\sqrt[4]{5}$ c. $2\sqrt[5]{6}$ d. $-5\sqrt[3]{4}$ e. Not real
f. $-3\sqrt[5]{2}$ g. $10\sqrt[3]{3}$ h. $3\sqrt[4]{7}$ i. $6\sqrt[3]{3}$ j. $2\sqrt[5]{10}$
k. $3\sqrt[3]{3}$ l. $5\sqrt[3]{2}$ m. $-2\sqrt[3]{7}$ n. $2\sqrt[4]{3}$ o. Not real
p. $4\sqrt[4]{3}$ q. $2\sqrt[5]{4}$ r. $3\sqrt[5]{5}$

“An educational system isn't worth a great deal if it teaches young people how to make a living but doesn't teach them how to make a life.”

– Unknown